MATH 2050 C Lecture 13 (Mar 1)

Reminder: Midterm this Thursday 8:30am - 10:15am

hard submission deadline

Please email me if you have any questions before / during the test.

Recall: "Subsequences"

Let $(x_n)_{n \in \mathbb{N}}$ be a seq. For any strictly increasing

$$n_1 < n_2 < n_3 < \cdots \quad \text{of natural numbers,}$$

we can form a subseq. $(x_{n_k})_{k \in \mathbb{N}}$

$$(x_{n_k})_{k \in \mathbb{N}} = (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$$

Thm: $(x_n) \to x \iff$ EVERY subseq $(x_{n_k}) \to x$

Remark: Hence, to show $(x_n) \not\to x$, we need to find

(i) $\exists$ subseq. (x_{n_k}) which divergent

(ii) two subseq. converging to different limits.

Thm: " (x_n) does NOT converge to x "

Caution: (x_n) may be converging to some $x' \neq x$

$\iff \exists\, \varepsilon_0 > 0$ AND a subseq. (x_{n_k}) s.t

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \forall k \in \mathbb{N}$$

Proof: From def$^{\underline{n}}$, "(x_n) does converge to x"

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon \quad \forall n \geq K$$

Taking the negation of the statement,

"(x_n) does <u>NOT</u> converge to x"

$$\Leftrightarrow \left[\begin{array}{c} \exists \varepsilon_0 > 0 \text{ s.t. } \forall K \in \mathbb{N}, \ \exists n_K \geq K \text{ s.t.} \\ |x_{n_K} - x| \geq \varepsilon_0 \end{array} \right] \quad \cdots (*)$$

Idea: take $K = 1, 2, 3, \ldots$ obtain $n_1, n_2, n_3, \ldots$

Want to define the subseq. $(x_{n_k})_{k \in \mathbb{N}}$

Caution: We may not have $n_1 < n_2 < n_3 < \cdots$

Further refinement: need to choose n_k's more carefully.

Do it one term at a time.

- For $K = 1$, $(*) \Rightarrow \exists n_1 \geq 1$ st $|x_{n_1} - x| \geq \varepsilon_0$
- For $K = n_1 + 1$, $(*) \Rightarrow \exists n_2 \geq n_1 + 1$ st.

$$|x_{n_2} - x| \geq \varepsilon_0$$

Repeat $\rightsquigarrow$ $(x_{n_k})_{k \in \mathbb{N}}$ st $|x_{n_k} - x| \geq \varepsilon_0$

$\blacksquare$

Recall: MCT: (x_n) bdd & monotone $\Rightarrow$ (x_n) convergent.

Q: What can we say if (x_n) is ONLY bdd?

Bolzano-Weierstrass Thm "BWT"

(x_n) bdd $\Rightarrow$ $\exists$ subseq. (x_{n_k}) which is convergent

Remark: $((-1)^n) = (x_n)$ has converging subseq.

$(1, 1, \dots) \to 1$ and $(-1, -1, -1, \dots) \to -1$.

which have different limits.

Proof: Our prove is based on the "Nested Interval Property" (NIP)

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$$

closed & bdd

[If, Length$(I_n) \to 0$, then $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.]

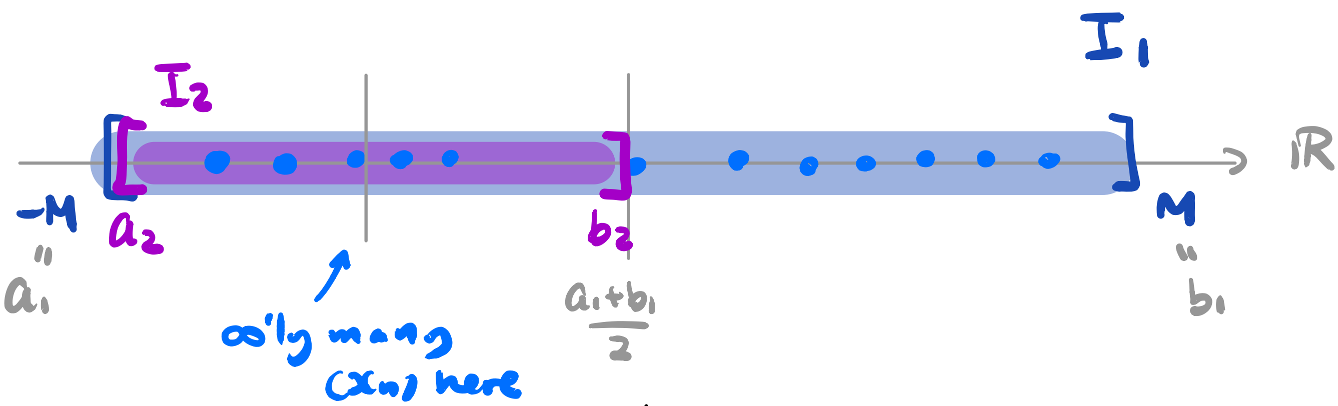
GOAL: Construct intervals I_n satisfying the hypothesis, using the "method of bisection".

Let (x_n) be a bdd seq, i.e. $\exists$ $M > 0$ s.t.

$$|x_n| \leq M \quad \forall n \in \mathbb{N}.$$

Define $I_1 := [a_1, b_1] = [-M, M]$

Notice that $x_n \in I_1 \quad \forall n \in \mathbb{N}$



Consider the midpoint $\frac{a_1+b_1}{2}$

<u>Case 1</u>: $[a_1, \frac{a_1+b_1}{2}]$ contains ∞'ly many terms of (x_n)

$\rightsquigarrow$ choose $I_2 := [a_2, b_2] = [a_1, \frac{a_1+b_1}{2}]$.

<u>Case 2</u>: Otherwise, choose $I_2 = [\frac{a_1+b_1}{2}, b_1]$.

Repeat, take a midpt $\frac{a_2+b_2}{2}$, choose $I_3 = [a_3, b_3]$

which is half of I_2 containing ∞'ly many terms of (x_n)

Inductively, we constructed a seq. of closed & bdd intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$ "nested"

s.t.
- each I_n contains <u>∞'ly many terms in (x_n)</u>
- $\text{Length}(I_n) = \frac{2M}{2^{n-1}} \longrightarrow 0$ as $n \to \infty$

Apply NIP $\Rightarrow$ $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$ ie $\lim(a_n) = \lim(b_n) = \xi$

$I_n = [a_n, b_n]$

Claim: $\exists$ subseq. $(x_{n_k}) \to \xi$

Pf: Pick any $n_1 \in \mathbb{N}$ s.t $x_{n_1} \in I_1$.

Then, pick $n_2 > n_1$ s.t $x_{n_2} \in I_2$, which is possible since I_2 contains ∞'ly many terms of (x_n).

Then, pick $n_3 > n_2$ s.t $x_{n_3} \in I_3$.

Inductively, we obtain a seq. $(x_{n_k})_{k\in\mathbb{N}}$ s.t $x_{n_k} \in I_k = [a_k, b_k]$ $\forall k \in \mathbb{N}$

ie $a_k \le x_{n_k} \le b_k$ $\forall k \in \mathbb{N}$

Since $\lim(a_k) = \lim(b_k) = \xi$, by Squeeze Thm, then $\lim_{k\to\infty} x_{n_k} = \xi$.

$\square$

We now give one application of BWT.

Prop: Suppose (x_n) is a bdd seq.

$\lim(x_n) = x \iff$ ANY convergent subseq. (x_{n_k}) has $\lim_{k\to\infty} x_{n_k} = x$

Proof: "$\Rightarrow$" trivial (done.)

"$\Leftarrow$" By contradiction. Suppose $(x_n) \not\to x$.

By Thm before, $\exists\, \varepsilon_0 > 0$ & a subseq (x_{n_k}) s.t.

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \forall k \in \mathbb{N} \cdots (\#)$$

Note that (x_n) bdd $\Rightarrow$ (x_{n_k}) bdd

By BWT, $\exists$ a further subseq $(x_{n_{k_\ell}})_{\ell \in \mathbb{N}}$

of $(x_{n_k})_{k \in \mathbb{N}}$ (which is a subseq of $(x_n)_{n \in \mathbb{N}}$)

which is convergent

By hypothesis, $\lim_{\ell \to \infty} x_{n_{k_\ell}} = x$ contradicting $(\#)$.

$\blacksquare$

Subsequential Limits: limsup & liminf

Q: Given a bdd seq. (x_n), what is

$$\mathcal{L} := \{ \ell \in \mathbb{R} \mid \exists \text{ subseq } (x_{n_k}) \text{ of } (x_n) \text{ s.t. } \lim (x_{n_k}) = \ell \} \ ?$$

Examples: If $\lim(x_n) = x$, then $\mathcal{L} = \{x\}$.

If $(x_n) = ((-1)^n)$, then $\mathcal{L} = \{1, -1\}$.

Note that since (x_n) is bdd.

$$BWT \implies \mathcal{L} \neq \phi$$

On the other hand, (x_n) bdd means that

$\exists M > 0$ s.t. $|x_n| \leq M \quad \forall n \in \mathbb{N}$.

$\Rightarrow$ If (x_{n_k}) is a converging subseq w. limit ℓ,

then $\quad -M \leq x_{n_k} \leq M \quad \forall k \in \mathbb{N}$

By Limit Thm. $\quad -M \leq \lim_{k\to\infty} x_{n_k} = \ell \leq M$

So, $\phi \neq \mathcal{L} \subseteq [-M, M]$ is a non-empty bdd subset of $\mathbb{R}$. By Completeness of $\mathbb{R}$, the inf and sup of $\mathcal{L}$ must exist in $\mathbb{R}$.

Def$^{\underline{n}}$:

$$\boxed{\begin{aligned} \limsup (x_n) &= \overline{\lim}\,(x_n) := \sup \mathcal{L} \\ \liminf (x_n) &= \underline{\lim}\,(x_n) := \inf \mathcal{L} \end{aligned}}$$

Examples: If $\lim(x_n) = x$, then

$$\overline{\lim}\,(x_n) = \underline{\lim}\,(x_n) = x = \lim(x_n).$$

If $(x_n) = ((-1)^n)$, then $\mathcal{L} = \{-1, 1\}$ hence

$$\overline{\lim}\,(x_n) = 1 \quad \text{and} \quad \underline{\lim}\,(x_n) = -1$$

Thm: Let (x_n) be a bdd seq. Define a new seq. (u_m) by $u_m := \sup\{x_n \mid n \geq m\}$, $m = 1, 2, 3, \dots$

THEN: (u_m) is a decreasing seq. with

$$\lim_{m \to \infty} u_m = \inf\{u_m \mid m \in \mathbb{N}\} = \overline{\lim}\,(x_n)$$

Exercise: Formulate & proof an analogous statement for $\underline{\lim}\,(x_n)$.

Proof: From the def.ⁿ of u_m,

$$(x_n) = (x_1, x_2, x_3, x_4, x_5, \dots, x_n, \dots)$$

$\sup = u_1$ $\quad$ $\sup = u_2$

(Recall: $S_1 \subseteq S_2 \Rightarrow \sup S_1 \leq \sup S_2$)

So, $\forall m \in \mathbb{N}$, $\{x_n \mid n \geq m\} \supseteq \{x_n \mid n \geq m+1\}$

take sup $\Rightarrow$ $\quad u_m \geq u_{m+1}$

So, (u_m) forms a decreasing seq.

Since (x_n) is bdd, (u_m) is also bdd.

By **MCT**, $\lim_{m\to\infty}(u_m) = \inf\{u_m \mid m \in \mathbb{N}\}$.

It remains to show

$$\lim_{m\to\infty}(u_m) = \inf\{u_m \mid m \in \mathbb{N}\} = \overline{\lim}\,(x_n)$$

Step 1: $\overline{\lim}\,(x_n) \leq \lim(u_m)$

By def^n, $\overline{\lim}\,(x_n) = \sup \mathcal{L}$. Take any $l \in \mathcal{L}$.

by def^n, $\exists$ subseq (x_{n_k}) of (x_n) s.t.

$$(x_{n_k}) \longrightarrow l \quad \text{as } k \to \infty$$

$\forall k \in \mathbb{N}$, $\quad x_{n_k} \leq u_{n_k} := \sup\{x_n \mid n \geq n_k\}$

take $k \to \infty$. $\quad l \leq \lim_{k\to\infty}(u_{n_k}) = \lim_{m\to\infty}(u_m)$

Step 2: $\overline{\lim}\,(x_n) \geq \lim(u_m)$.

<u>Claim</u>: $\lim(u_m) \in \mathcal{L}$

We have to find a subseq (x_{n_k}) of (x_n) s.t. $(x_{n_k}) \to \lim(u_m)$.

- Choose $n_1 \geq 1$ s.t.

$$u_1 - 1 < x_{n_1} \leq u_1 := \sup\{x_n \mid n \geq 1\}$$

- Choose $n_2 > n_1$ s.t.

$$u_{n_1+1} - \frac{1}{2} < x_{n_2} \leq u_{n_1+1} := \sup\{x_n \mid n \geq n_1+1\}$$

Repeat inductively, we choose $n_1 < n_2 < n_3 < \cdots$

s.t. $$u_{n_k+1} - \frac{1}{k+1} < x_{n_k} \leq u_{n_k+1} \quad \forall k \in \mathbb{N}$$

Take $k \to \infty$, by Squeeze Thm.

$$\lim(u_m) = \lim_{k\to\infty}(x_{n_k}) \in \mathcal{L}$$

$\square$